113 Class Problems: Finitely Generated Abelian Groups

1. (a) Let $G = (\mathbb{C} \setminus \{0\}, \times)$. Give an explicit description of tG. Is G finitely generated?

   (b) Let $G = (\mathbb{Q} \setminus \{0\}, \times)$. Give an explicit description of tG. Is G finitely generated?

   Solutions:

a) $tG = \{x \in \mathbb{C} \setminus \{0\} \mid \exists n \in \mathbb{N} \text{ such that } x^n = 1\} = \{e^{2\pi i \alpha} \mid \alpha \in \mathbb{Q}\}$

$|tG| = \infty \Rightarrow G$ <u>not</u> f.g.

b) $tG = \{x \in \mathbb{Q} \setminus \{0\} \mid \exists n \in \mathbb{N} \text{ such that } x^n = 1\} = \{\pm 1\}$

Unfortunately, tG finite $\not\Rightarrow$ G f.g.

Instead $\mathbb{Q} \setminus \{0\}$ is <u>not</u> f.g. because of the FTOA and the infinitude of primes

2. Let $G = \mathbb{Z}^3 \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5^2\mathbb{Z}$.

   (a) Prove that G is a finitely generated group.

   (b) Give an explicit description of tG.

   (c) What is the rank of G?

   Solutions:

a)
$$G = gp\big((1,0,0,[0]_5,[0]_{5^2}),\ (0,1,0,[0],[0]_{5^2}),\ (0,0,1,[0]_5,[0]_{5^2}),\ (0,0,0,[1]_5,[0]_{5^2}),\ (0,0,0,[0]_5,[1]_{5^2})\big)$$

b) $tG = \{(0,0,0,[x]_5,[y]_{5^2}) \mid x, y \in \mathbb{Z}\} \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5^2\mathbb{Z}$

c) $\text{Rank}(G) = 3$